Applied Mathematics & Artifcial Intelligence: second-stage sample tasks

1 Linear algebra and analytic geometry

1. Compute the distance from the point $A = (1; 5; 1)$ to the plane $2x + y - 2z + 5 = 0$.

   Answer: 0.

   Maximum score: 4

2. Consider the polynomial $p(x) = (x^2 - 3x + 2) \cdot q(x)$, such that $p(x), q(x) \in \mathbb{R}[x]$. Which of the following statements can be correct: $p'(1) = 0$; $p'(1) \neq 0$; $p'(x) = (2x - 3) \cdot q'(x)$; $p'(\frac{3}{2}) = 0$?

   (a) $p'(1) = 0$
   (b) $p'(1) \neq 0$
   (c) $p'(\frac{3}{2}) = 0$
   (d) $p'(x) = (2x - 3) \cdot q'(x)$

   Answer: (a), (b), (c)

   Maximum score: 2

3. Consider $U, V \leqslant \mathbb{R}^4$: $U = \langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \rangle$ and $V = \langle \begin{pmatrix}0\\0\\1\\1\end{pmatrix}; \begin{pmatrix}0\\0\\-1\\1\end{pmatrix} \rangle$. Compute $\dim(\mathrm{U}^\perp + \mathrm{V}^\perp)$.

   Answer: 3.

   Maximum score: 2

4. Let $\varepsilon_0, \ldots, \varepsilon_{99}$ be a full set of complex roots of unity of degree 100. Compute $\sum_{k=0}^{99} \varepsilon_k$.

   Answer: 0.

   Maximum score: 8

   Solution:

   Numbers $\varepsilon_0, \ldots, \varepsilon_{99}$ are a full set of complex roots of polynomial $p(z) = z^{100} - 1$. By Viete's theorem, $\sum_{k=0}^{99} \varepsilon_k = 0$.

5. Consider the matrix $\begin{pmatrix}1 & 2 & -1 & 2\\0 & -1 & 5 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{pmatrix}$. What is the biggest number of linearly independent eigenvectors with the same eigenvalue for this matrix?

   Answer: 2.

   Maximum score: 2

2 Real and complex analysis

1. Compute the integral $\int_0^{\pi} 7\tan(\frac{x}{4})dx + \int_0^{7} 4\arctan(\frac{y}{7})dy$.

   Answer: 7π.

   Maximum score: 4

2. Compute $\lim\limits_{x\to+\infty}(\sqrt{x^2-x+1}-\sqrt{x^2+3x-3}.)$

   Answer: -2.

   Maximum score: 2

3. Select for which of the following functions the Taylor polynomial with degree 1 at $x_0 = 0$ is $T(x) = 2x + 1$?

   (a) $(x+1)^2$
   (b) $\cos(2x)$
   (c) $\sin(2x+1)$
   (d) e^{2x}

   Answer: (a), (d)

   Maximum score: 2

4. Which domains can be a conformal image of a disk with a cut?

   (a) circle
   (b) plane
   (c) annulus
   (d) rectangle

   Answer: (a), (b), (d)

   Maximum score: 4

5. Find the maximum of the function $f(x, y, z) = 7x + 4y + 4z$ on the sphere $x^2 + y^2 + z^2 = 81$.

   Answer: 81.

   Maximum score: 2

3 Differential equations

1. Consider the differential equation $y'' + 3y = \cos^2(x)$. Compute $\int\limits_0^{2\pi} y(x)\cos(2x)dx$.

   Answer: $-\frac{\pi}{2}$.

   Maximum score: 2

2. Consider the system of differential equation

$$\begin{cases} \dot{x} = y + 2z; & x(0) = 1 \\ \dot{y} = -x + 3z; & y(0) = 2 \\ \dot{z} = -2x - 3y; & z(0) = 3 \end{cases}$$

   Compute $x^2(1) + y^2(1) + z^2(1)$.

   Answer: 14.

   Maximum score: 2

3. Assume that the function

$$G(x,y) = \begin{cases} x(1-y), & 0 \leqslant x \leqslant y \leqslant 1; \\ (1-x)y, & 0 < y < x < 1 \end{cases}$$

   is the Green function of a Sturm-Liouville problem on the interval $[0, 1]$ with Dirichlet boundary conditions. For which Sturm-Liouville operators with Dirichlet boundary conditions is the function G a Green function?

   (a) $-u''$;

(b) $-2u''$,

(c) $-u'' + u$,

(d) $-u'' - u$,

(e) $-2u'' + u$;

(f) $-u'' - 2u$.

Answer: (a).

MAXIMUM SCORE: 4

4. Which systems of functions can be a fundamental system of solutions of linear differential equation of order 3 with constant coefficients and the right-hand side xe^x?

   (a) $\{1, e^x, xe^x\}$;

   (b) $\{1, e^x, e^{3x}\}$;

   (c) $\{x, e^x, e^{3x}\}$;

   (d) $\{e^x, e^{-x}\}$;

   (e) $\{e^{3x}; xe^{3x}; e^{-3x}\}$

   Answer: (a), (b), (e).

   MAXIMUM SCORE: 2

5. Find the minimum of the functional

$$\int_0^1 ((u'')^2 - 2u)\mathrm{dx}$$

   such that $u(0) = u(1) = u'(0) = u'(1) = 0$.

   Answer: $-\frac{1}{720}$

   MAXIMUM SCORE: 5

4 Theory of probability and mathematical statistics

1. Two machines produce details. The probability of manufacturing a standard part by the first automaton is 0.85; by the second, 0.9. The performance of the first machine is twice that of the second. A worker took a part at random, and it turned out to be standard. What is the probability that this part was made by the first machine? Round your answer to three decimal places.

   Answer: 0.654

   MAXIMUM SCORE: 2

2. A total of 4000 serviceable products have been sent to the warehouse. The probability that the product will be damaged in transit is 0.0005. Find the probability that out of 4000 items, no more than 2 damaged items will arrive at the warehouse. Round your answer to two decimal places.

   Answer: 0.68

   MAXIMUM SCORE: 2

3. The two-dimensional random variable (ξ, η) is given by the joint distribution density

$$f_{\xi,\eta}(x, y) = C\, e^{-x^2+2xy-y^2+1}, \quad x, y \in \mathbb{R}^2,$$

   where $C \in \mathbb{R}$ is a parameter.

   Find the conditional expectation $\mathsf{E}(\xi \mid \eta = 1)$ of the random variable ξ given that $\eta = 1$.

   Answer: 1

   MAXIMUM SCORE: 4

4. A fair six-sided die is tossed 20 times. Find the correlation coefficient between the number of occurrences of two and the number of occurrences of three on the dice.

   Answer: $-\frac{1}{5}$

   MAXIMUM SCORE: 9

   Solution:

   Let us solve the problem in a general way. Let there be n rolls of the die.

   Denote by $\xi_i,\ i = 1, 2, \ldots, 6$, a random value equal to the number of occurrences of the face with i points during n rolls of the die. Let us calculate the covariance $cov(\xi_2, \xi_4)$. Each of the random variables ξ_i has a binomial distribution with parameters n and $1/6$, so $\mathsf{E}(\xi_i) = n/6$, $\mathsf{D}(\xi_i) = 5n/36$.

   Further note that $\xi_1 + \ldots + \xi_6 = n$. Due to the symmetry of the cube, the mathematical expectations $\mathsf{E}(\xi_2\,\xi_1)$, $\mathsf{E}(\xi_2\,\xi_3)$, $\mathsf{E}(\xi_2\,\xi_4)$, $\mathsf{E}(\xi_2\,\xi_5)$, $\mathsf{E}(\xi_2\,\xi_6)$ are the same. Moreover,

$$\mathsf{E}(\xi_2\,\xi_2) = \mathsf{E}(\xi_2^2) = \mathsf{D}(\xi_2) + \mathsf{E}^2(\xi_2) = 5n/36 + n^2/36.$$

   Let us calculate $\mathsf{E}(\xi_2\,(\xi_1 + \ldots + \xi_6)$. On the one hand, it is equal to

$$\mathsf{E}(\xi_2\,(\xi_1 + \ldots + \xi_6)) = \mathsf{E}(\xi_2)n = n^2/6,$$

   on the other,

$$\mathsf{E}(\xi_2\,(\xi_1 + \ldots + \xi_6)) = \mathsf{E}(\xi_2^2) + 5\mathsf{E}(\xi_2\,\xi_4) = 5n/36 + n^2/36 + 5\mathsf{E}(\xi_2\,\xi_4).$$

   Thus,

$$n^2/6 - 5n/36 - n^2/36 = 5\mathsf{E}(\xi_2\,\xi_4).$$

$$\mathsf{E}(\xi_2\,\xi_4) = \frac{n^2 - n}{36}.$$

   Therefore, the desired correlation coefficient is equal to

$$r(\xi_2,\,\xi_4) = \frac{cov(\xi_2, \xi_4)}{\sqrt{\mathsf{D}(\xi_2)\mathsf{D}(\xi_2)}} = \frac{\mathsf{E}(\xi_2\,\xi_4) - \mathsf{E}(\xi_2)\mathsf{E}(\xi_4)}{5n/36} = \frac{(n^2 - n)/36 - n^2/36}{5n/36} = -\frac{1}{5}.$$

   The resulting correlation coefficient does not depend on n.

5. A random odd binary number less or equal to 2^{10} is taken. Find the probability of its binary notation containing precisely five symbols "1".

   Answer: $\frac{63}{256}$.

   MAXIMUM SCORE: 2

5 Mathematical foundations of artificial intelligence

1. Consider the classification problem. Here are the class labels for the test dataset:

$$y_{\text{true}} = (2, 2, 2, 1, 1, 0, 2, 1, 0, 2)^T.$$

   A machine learning algorithm predicts the following labels:

$$y_{\text{pred}} = (0, 0, 0, 2, 1, 1, 2, 1, 2, 2)^T.$$

   Compute the mutual information score between y_{true} и y_{pred}. Round your answer to two decimal places.

   Answer: 0.42

   MAXIMUM SCORE: 2

2. A problem of linear regression is considered. Here is the feature matrix of a dataset:

$$X = \begin{pmatrix} 1 & -3 & -4 & -2 & -3 \\ -2 & 1 & -3 & 2 & -1 \end{pmatrix}^T.$$

Consider the following values of the target function:

$$y = (-2.6, 2.2, 7.9, -0.6, 4.2)^T.$$

Train the linear regression model using the least squares method. Compute the sum of squared coefficients in the model function (including the intercept!). Round your answer to two decimal places.

Answer: 11.10

MAXIMUM SCORE: 2

3. Let us introduce the following setting. Denote by CNN(in, out, ker) a convolutional layer with the in input channels, out output channels and ker standing for the convolutional kernel. Denote by FC(in, out) a linear layer in input neurons and the in out output neurons. Furthermore, let us denote by MP($size$) a maxpooling layer of a size equal to the $size$ value, $padding = 0$, $stride = size$. Also, denote by AVP the global average pooling layer (the layer that turns a tensor of form $(, , h, w)$ into a tensor of form $(, , 1, 1)$ by computing the average value across the (h, w) kernel). Finally, denote by FL the standard flattening operation. Therefore, the pair AVP → FL turns feature maps into a 1-d vector.

Consider the model with the following diagram:

$$\text{CNN}(3,16,3) \to \text{CNN}(16,16,1) \to \text{MP}(2) \to \text{CNN}(16,32,3) \to \text{CNN}(32,32,3) \to \text{MP}(2) \to \text{CNN}(32,32,1) \to$$
$$\to \text{AVP} \to \text{FL} \to \text{FC}(32,24) \to \text{FC}(24,10).$$

Compute the number of the weights in this model.

Answer: 16706

MAXIMUM SCORE: 3

4. Draw a directed graph according to the following diagram (an arrow denotes directed pair):

$$A \to B \to C, A \to D \to E, F \to D, G \to C.$$

Consider the joint distribution of variables $A, \dots, G$. This distribution is true to the structure of the graph considered (e.g. each conditional independence true to the distribution can be decoded by the graph's d-separation properties and vice versa). Which of the following propositions about conditional independencies is true?

(a) A, E conditionally independent given D
(b) A, G conditionally independent given C
(c) A, D conditionally independent given B
(d) A, F conditionally independent given D

Answer: (a)

MAXIMUM SCORE: 4

5. Consider the problem of regression. The model is a feedforward neural network. Consider the following input:

$$X = (0.0, -1.0, -1.0)$$

and the target function

$$y = (1.0).$$

Denote by FC(in, out) a fully connected linear layer with in inputs and out outputs. Let us denote by ReLU the rectified linear unit (ReLU) activation function. Consider the following neural network architecture:

$$\text{FC}(3,2) \to \text{ReLU} \to \text{FC}(2,1).$$

Let us initiate the parameters of the above-mentioned network by the following values:

$$W_1 = \begin{pmatrix} -1.0 & -2.0 & -1.0 \\ -2.0 & -1.0 & -1.0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} -1.0 & 1.0 \end{pmatrix},$$

$$b_1 = \begin{pmatrix} -2.0 & -2.0 \end{pmatrix}^T,$$

$$b_2 = (1.0).$$

Here, we denote by W_i the weights of the i-th layer, we use the same notation for the corresponding biases b_i. Compute the model prediction $\widehat{y}$. Consider $L = (\widehat{y} - y)^2$ as the model's loss function. Next, do a step of backpropagation algorithm and compute the values of the $\frac{\partial L}{\partial W_1}$. Sum up the squared cell values of this matrix. Round your answer to two decimal places.

Answer: 8.0

MAXIMUM SCORE: 4

6 Discrete Mathematics

1. Consider a graph $G(V, E)$, where V is a set of vertices, E is a set of edges. Consider spanning tree T_G for graph G with 10 edges. Which of the following statements can be true of G?

   (a) $|E| = 10$
   (b) $|E| = 45$
   (c) $|E| = 55$
   (d) $|E| = 60$
   (e) $|V| = 10$
   (f) $|V| = 11$
   (g) $|E| < \frac{|V|}{2}$
   (h) $|E| = |V|$

   Answer: (a), (b), (c), (f), (h)

   MAXIMUM SCORE: 2

2. Consider coins with values 3 and 11 racoons. How many coins do you need to have precisely 49 racoons? Compute the minimal quantity.

   Answer: 11.

   MAXIMUM SCORE: 2

3. Let $a, b \in \mathbb{N}$, $\text{GCD}(a, b) = 5$. Compute all variants of the value $\text{GCD}(6a, b)$.

   (a) $\frac{5}{6}$
   (b) 1
   (c) 5
   (d) 10
   (e) 11
   (f) 15
   (g) 30
   (h) 60

   Answer: (c), (d), (f), (g)

   MAXIMUM SCORE: 4

4. Find full sets of polynomials $P(x) \in \mathbb{Z}_7[x]$, such that $(x-1) \cdot P(x) \equiv 1 \ ((\text{mod}) \ (\mathrm{x}^2+3\mathrm{x}+2)$.

   Answer: $P(x) = x + 4 + h(x) \cdot (x^2 + 3x + 2), \ h(x) \in \mathbb{Z}_7[x]$ (or any equivalent)

   MAXIMUM SCORE: 9

   Solution:

   By definition of equivalence $(x-1) \cdot P(x) \equiv 1 \ ((\text{mod}) \ (\mathrm{x}^2+3\mathrm{x}+2) \Leftrightarrow (\mathrm{x}-1) \cdot \mathrm{P(x)} + \mathrm{y(x)} \cdot (\mathrm{x}^2+3\mathrm{x}+2) = 1, \ \mathrm{y(x)} \in \mathbb{Z}_7[\mathrm{x}]$. This equation is a Diophantine equation on $(P(x), y(x))$ over the Euclidean ring $\mathbb{Z}_7[x]$. Using any methods, find the specific solution: $\begin{cases} P(x) = x + 4 \\ y(x) = -1 \end{cases}$. The solution of the corresponding homogeneous equation: $\begin{cases} P_0(x) = h(x) \cdot (x^2 + 3x + 2) \\ y_0(x) = -h(x) \cdot (x - 1) \end{cases}$, where $h(x) \in \mathbb{Z}_7[x]$. Thus, the full sought-for set of the polynomials is $P(x) = x + 4 + h(x) \cdot (x^2 + 3x + 2), \ h(x) \in \mathbb{Z}_7[x]$.

5. Consider the polynomial $p(x) = (1 + x^2 + x^3)^{20}$. Compute the coefficient for x^{12}.

   Answer: 198645.

   MAXIMUM SCORE: 2